MIXED-MODE OSCILLATIONS IN FITZHUGH NAGUMO MODEL

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Introduction

We study the stochastic FitzHugh-Nagumo equation, modelling the dynamics of neuronal action potential in the axon of a neuron.

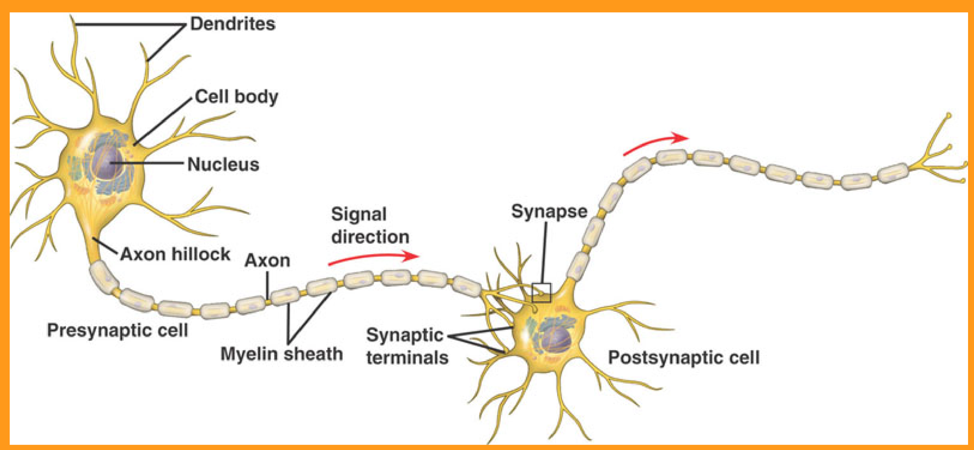

FIGURE 1: The structure of a neuron

The general model is a slow-fast system of stochastic differential equation:

$$\begin{cases} \varepsilon dx_t = \left(x_t - \dfrac{x_t^3}{3} + y_t\right) dt + \sqrt{\varepsilon}\sigma_1 \, dW_t^{(1)} \\ dy_t = (a - x_t)\, dt + \sigma_2 \, dW_t^{(2)} \end{cases} \tag{1}$$

Here x is the fast variable and represents the membrane potential, y is the slow variable, a is a positive parameter, ε is a small positive parameter ($\varepsilon << 1$), σ_1 and σ_2 are small positive parameters ($\sigma << 1$) representing the noise amplitude of the independant Brownian Motions $W_t^{(1)}$ and $W_t^{(2)}$.

Deterministic equation

We consider the deterministic equation associated to the SDE (1):

$$\begin{cases} \varepsilon \dot{x} = x - \dfrac{x^3}{3} + y \\ \dot{y} = a - x \end{cases} \tag{2}$$

First of all, we study the equilibrium point P of the equation (2) given by $(x^*, y^*) = (a, a^3 - a)$. It is a Hopf bifurcation point. Let

$$\delta = (3a^2 - 1)/2, \tag{3}$$

such that δ is small if the equilibrium point is near the Hopf bifurcation point. We have three cases:

- if $\delta > \sqrt{\varepsilon}$: two real negative eigenvalues: P is a stable node.
- if $0 < \delta < \sqrt{\varepsilon}$: two real eigenvalues with one positive: P is a stable focus.
- if $-\sqrt{\varepsilon} < \delta < 0$: two complex eigenvalues with real part negative: P is an unstable focus and we have a limit cycle.

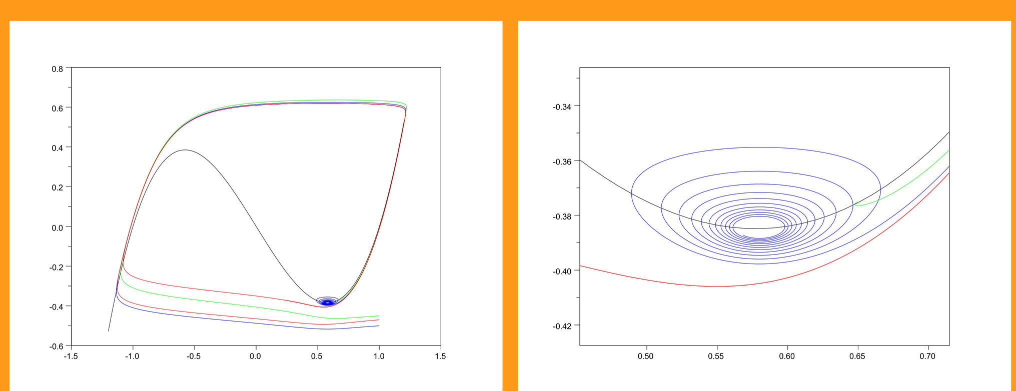

FIGURE 2: Three orbits of the deterministic FitzHugh-Nagumo equations for $\varepsilon = 0.05$

Spikes distribution

Now we add noise to the equation (2), we have four regimes:

- unstable focus : **loops near the limit cycle**.
- stable node or focus :
  - weak noise : **loops around the fixed point**.
  - intermediate noise : **loops around the fixed point and exit** to loop on the limit cycle.
  - strong noise : **loop near the limit cycle**.

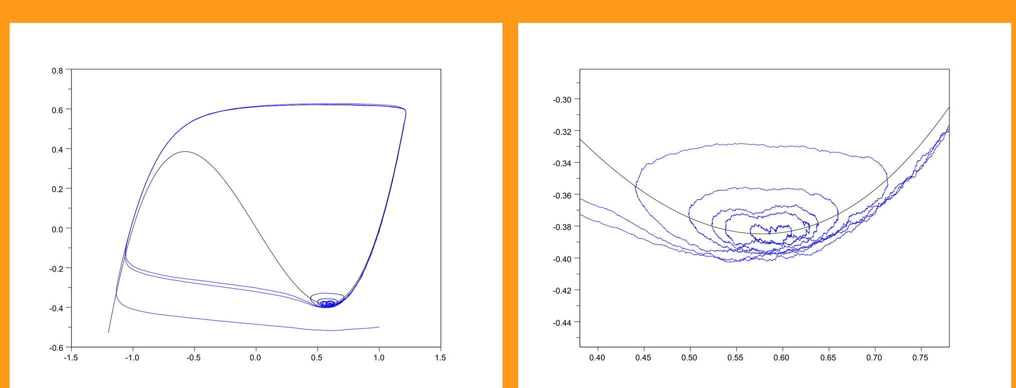

FIGURE 3: An orbit of the stochastic FitzHugh-Nagumo equations

Finally, we fix a and we plot the membrane potential x in function of the time t. We observe three different main regimes following the values of δ and σ:

- **numerous and regular spikes** : the trajectory stay only a short time around the equilibrium point before exiting.
- **spike or a cluster of spikes** from time to time. That means the trajectory stay some times around the equilibrium point before exiting and when it comes back, it can sometimes exit quickly
- **rare isolated spikes**.

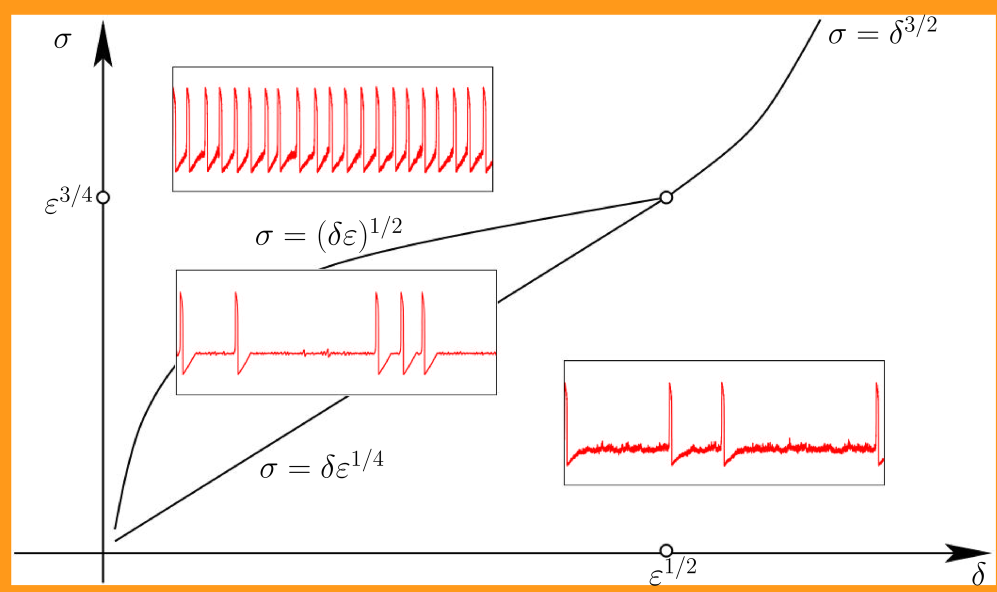

FIGURE 4: Phase diagram of the stochastic FitzHugh-Nagumo equations (see [2])

We want to study the probability distribution of small amplitude oscillation (SAO) betwenn two spikes in these different regimes.

Number of SAOs

We first define the integer-valued random variable N, couting the number of small-amplitude oscillations between two consecutive spikes.

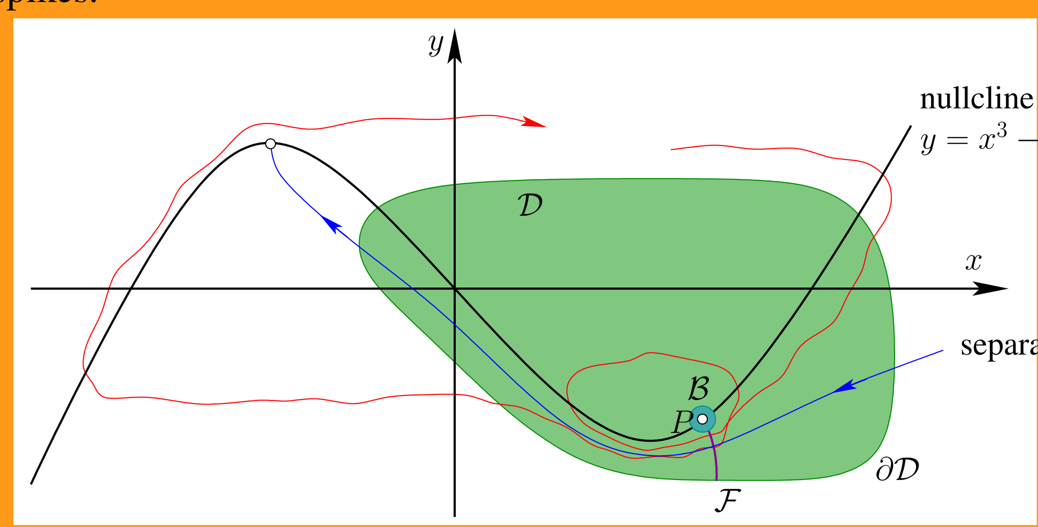

FIGURE 5: Definition of the number N of SAOs. Here $N = 2$

Let $\mathcal{D} \subset \mathbb{R}^2$, a bounded set containing the stationary point P and a piece of separatrix. If the sample path (x_t, y_t) leave $\mathcal{D}$, we consider we have a spike. A simple **definition of** N is the **number of times the sample path turn around** P **before leaving** $\mathcal{D}$. Let $(R_1, R_2, \dots, R_N)$ the successive intersections of the path with $\mathcal{F}$ separated by rotation around P. It ends with the exit from $\mathcal{D}$. The sequence $(R_n)_n$ forms a substochastic Markov chain with kernel

$$K(R, A) = \mathbb{P}\{R_{n+1} \in A \mid R_n = R\},\ R \in \mathcal{F},\ A \subset \mathcal{F} \tag{4}$$

The kernel K admits a principal eigenvalue λ_0. There exists a probability measure π_0 such that $\pi_0 K = \lambda_0 K$.
Our first main result gives qualitative properties of the distribution of N valid in all parameter regimes.

Theorem 1 (*General properties of N*)

Assume that $\sigma_1, \sigma_2 > 0$. Then for any initial distribution μ_0 of R_0 on the curve $\mathcal{F}$,

- *the kernel K admits a quasi-stationary distribution π_0;*
- *the associated principal eigenvalue $\lambda_0 = \lambda_0(\varepsilon, \delta, \sigma_1, \sigma_2)$ is strictly smaller than 1;*
- *the random variable N is almost surely finite;*
- *the distribution of N is "asymptotically geometric", that is,*

$$\lim_{n\to\infty} \mathbb{P}^{\mu_0}\{N = n + 1 \mid N > n\} = 1 - \lambda_0\,; \tag{5}$$

- *$\mathbb{E}^{\mu_0}\left\{r^N\right\} < \infty$ for $r < 1/\lambda_0$ and thus all moments $\mathbb{E}^{\mu_0}\left\{N^k\right\}$ of N are finite.*

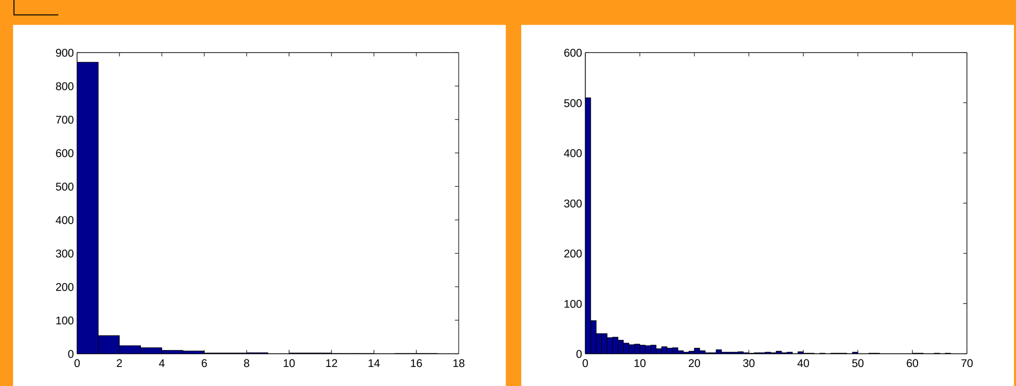

FIGURE 6: Histograms of the distributions of the SAO number N for $\tilde{\mu}/\tilde{\sigma} = -0.5$ and 0.1

If the initial distribution μ_0 is equal to π_0, the random variable R_n has the law $\mu_n = \lambda_0^n \pi_0$, and N follows an exponential law of parameter $1 - \lambda_0$:

$$\mathbb{P}^{\pi_0}\{N = n\} = \lambda_0^{n-1}(1 - \lambda_0) \quad \text{and} \quad \mathbb{E}^{\pi_0}\{N\} = \frac{1}{1 - \lambda_0}\,. \tag{6}$$

In general, however, the initial distribution μ_0 after a spike will be far from the QSD π_0, and thus the distribution of N will only be asymptotically geometric.

Theorem 2 (*Weak-noise regime*)

Assume that ε and $\delta/\sqrt{\varepsilon}$ are sufficiently small. Then there exists a constant $\kappa > 0$ such that for $\sigma_1^2 + \sigma_2^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$, the principal eigenvalue λ_0 satisfies

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma_1^2 + \sigma_2^2}\right\}. \tag{7}$$

Furthermore, for any initial distribution μ_0 of incoming sample paths, the expected number of SAOs satisfies

$$\mathbb{E}^{\mu_0}\{N\} \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma_1^2 + \sigma_2^2}\right\}. \tag{8}$$

Here $C(\mu_0)$ is the probability that the incoming path hits $\mathcal{F}$ above the separatrix.

Proposition

Let z_T^1 the distance to sepratrix (linearised) and $2L^2 = \gamma|\log(c_-\tilde{\mu})|$ for some $\gamma, c_- > 0$. Then for any H,

$$\mathbb{P}\{z_T^1 \leqslant -H\} = \Phi\left(-\pi^{1/4}\frac{\tilde{\mu}}{\tilde{\sigma}}\left[1 + \mathcal{O}\left((H + z_0)\tilde{\mu}^{\gamma-1}\right)\right]\right), \tag{9}$$

where $\tilde{\sigma}^2 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 = 3\varepsilon^{-3/2}(\sigma_1^2 + \sigma_2^2)$, $\tilde{\mu} = \delta/\varepsilon - \tilde{\sigma}_1^2$, and $\Phi(x) = \int_{-\infty}^x e^{-u^2/2} 6u/\sqrt{2\pi}$ is the distribution function of the standard normal law.

Choosing γ large enough, we expect that

$$1 - \lambda_0 \simeq \Phi\left(-\pi^{1/4}\frac{\tilde{\mu}}{\tilde{\sigma}}\right) = \Phi\left(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right). \tag{10}$$

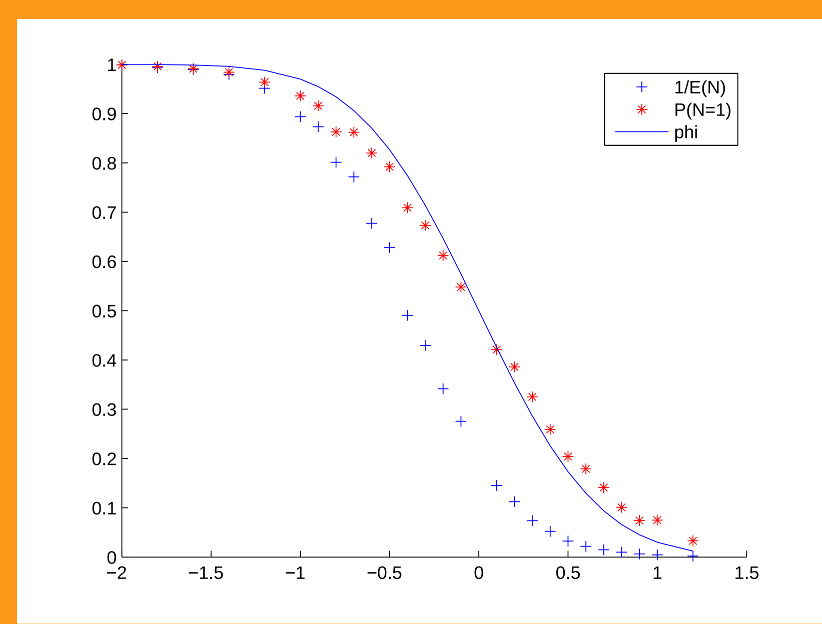

FIGURE 7: Comparison of $\Phi\left(-\pi^{1/4}\tilde{\mu}/\tilde{\sigma}\right)$ with $\mathbb{P}\{N = 1\}$ and $1/\mathbb{E}\{N\}$.

We can identify three regimes, depending on the value of $\tilde{\mu}/\tilde{\sigma}$:

1. **Weak noise :** $\tilde{\mu} \gg \tilde{\sigma}$, which in original variables translates into $\sqrt{\sigma_1^2 + \sigma_2^2} \ll \varepsilon^{1/4}\delta$, λ_0 is exponentially close to 1, and thus spikes are separated by long sequences of SAOs.
2. **Strong noise :** $\tilde{\mu} \ll -\tilde{\sigma}$, which implies $\mu \ll \tilde{\sigma}^2$, and in original variables translates into $\sqrt{\sigma_1^2 + \sigma_2^2} \gg \varepsilon^{3/4}$. Then λ_0 is exponentially small, of order $e^{-(\sigma_1^2+\sigma_2^2)/\varepsilon^{3/2}}$. With high probability, no complete SAO between consecutive spikes, i.e., the neuron is spiking repeatedly.
3. **Intermediate noise :** $|\tilde{\mu}| = \mathcal{O}(\tilde{\sigma})$, which translates into $\varepsilon^{1/4}\delta \leqslant \sqrt{\sigma_1^2 + \sigma_2^2} \leqslant \varepsilon^{3/4}$. Then the mean number of SAOs is of order 1. In particular, when $\sigma_1 = \sqrt{\varepsilon\delta}$, $\tilde{\mu} = 0$ and thus λ_0 is close to $1/2$.

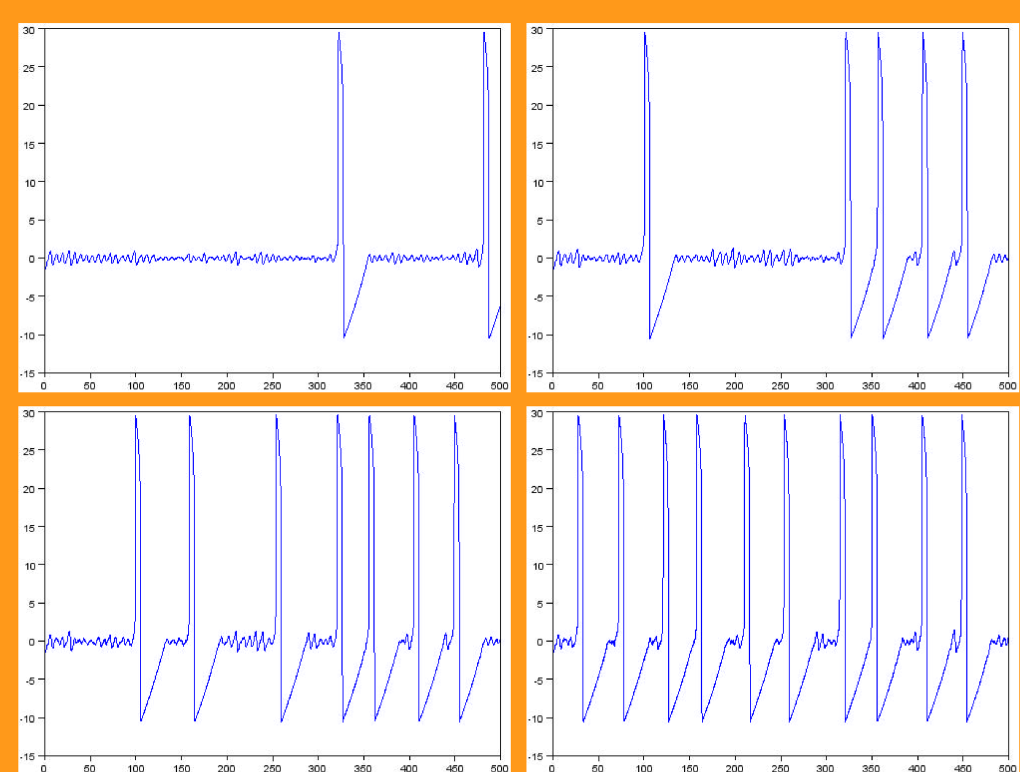

FIGURE 8: Examples of times series of (t, x_t)

The transition from weak to strong noise is gradual. There is no clear-cut transition at $\sigma_1 = \sqrt{\varepsilon\delta}$, the only particularity of this parameter value being that λ_0 is close to $1/2$.

References

[1] N. BERGLUND and D. LANDON Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model, Preprint, arXiv:1105.1278, 2011.

[2] C. MURATOV and E. VANDEN-EIJNDEN. Noised-induced mixed-mode oscillations in a relaxation oscillator near the onset of a limit circle, *Chaos*, 18, 2008.