SPIKES PROBABILITY DISTRIBUTION IN FITZHUGH-NAGUMO MODEL

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Introduction

We present here some results around the FitzHugh-Nagumo equation. The general model is a slow-fast system of stochastic differential equation:

$$\begin{cases} \varepsilon dx_t = \left(x_t - \frac{x_t^3}{3} + y_t\right) dt + \sqrt{\varepsilon}\sigma\, dW_t \\ dy_t = (\alpha - \beta x_t - \gamma y_t)\, dt \end{cases} \qquad \text{(FHN)}$$

Here x is the fast variable and represents the membrane potential, y is the slow variable, α, β and γ are postive parameters, ε is a small positive parameter ($\varepsilon << 1$), σ is a small positive parameter ($\sigma << 1$) representing the noise amplitude of the Brownian Motion W_t.

In a first part, we begin studying the case of deterministic equation associated to (FHN) with $\beta = 1$ and $\gamma = 0$, following the value of the parameter α.

In a second part, we give numerical simulations on the solution of the equation (FHN) with $\beta = 1$ and $\gamma = 0$, function of the values of α and σ.

In a third part, we consider the equation (FHN) with $\beta = 0$ and $\gamma = 1$. This equation can be reduced to one-dimensional ODS. We study this ODS in the neighborhood of the equilibrium point.

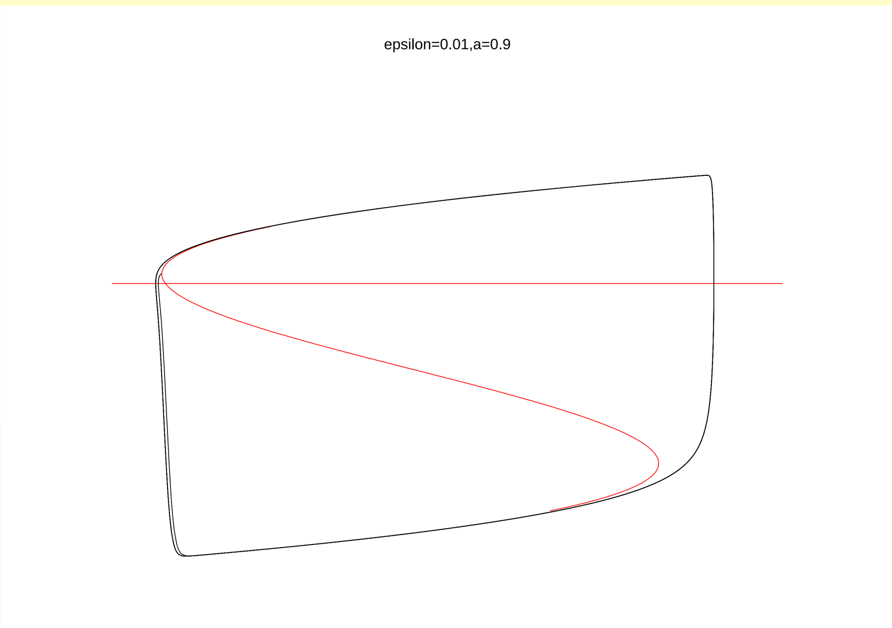
Deterministic equation

We consider the deterministic equation associated to the SDE (FHN) in the case $\beta = 1$ and $\gamma = 0$:

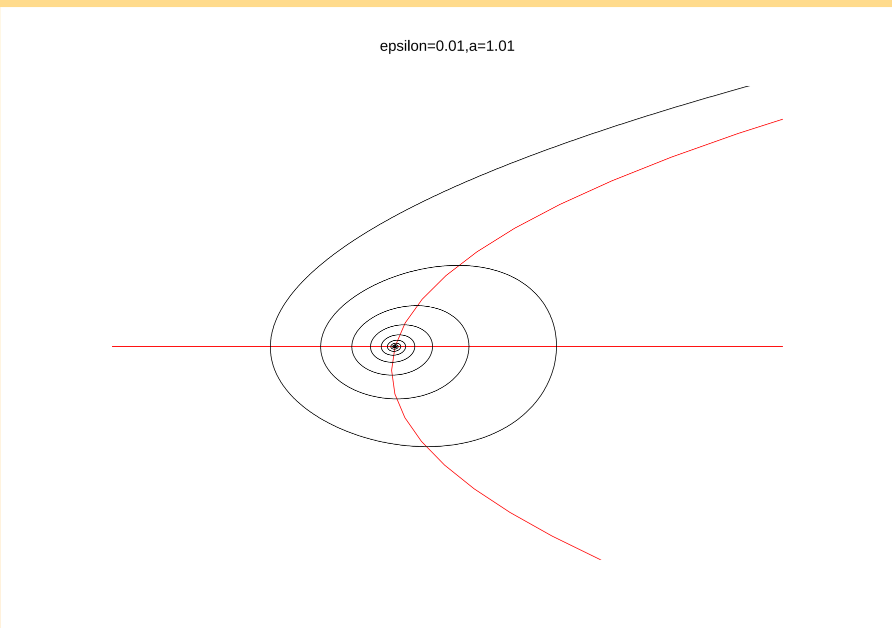
$$\begin{cases} \varepsilon \dot{x} = x - \frac{x^3}{3} + y \\ \dot{y} = \alpha - x \end{cases} \qquad (1)$$

First of all, we study the equilibrium point of the equation (1) given by $(x^*, y^*) = (\alpha, \frac{\alpha^3}{3} - \alpha)$. It is a Hopf bifurcation point. We have two cases:

- if $\alpha < \alpha_*$, the Jacobian matrix has two real eigenvalues. One of them is positive and we have a **stable node**.



- if $\alpha \geq \alpha_*$, the Jacobian matrix has two complex eigenvalues. The real part is negative and we have a **stable focus**.



Spikes distribution

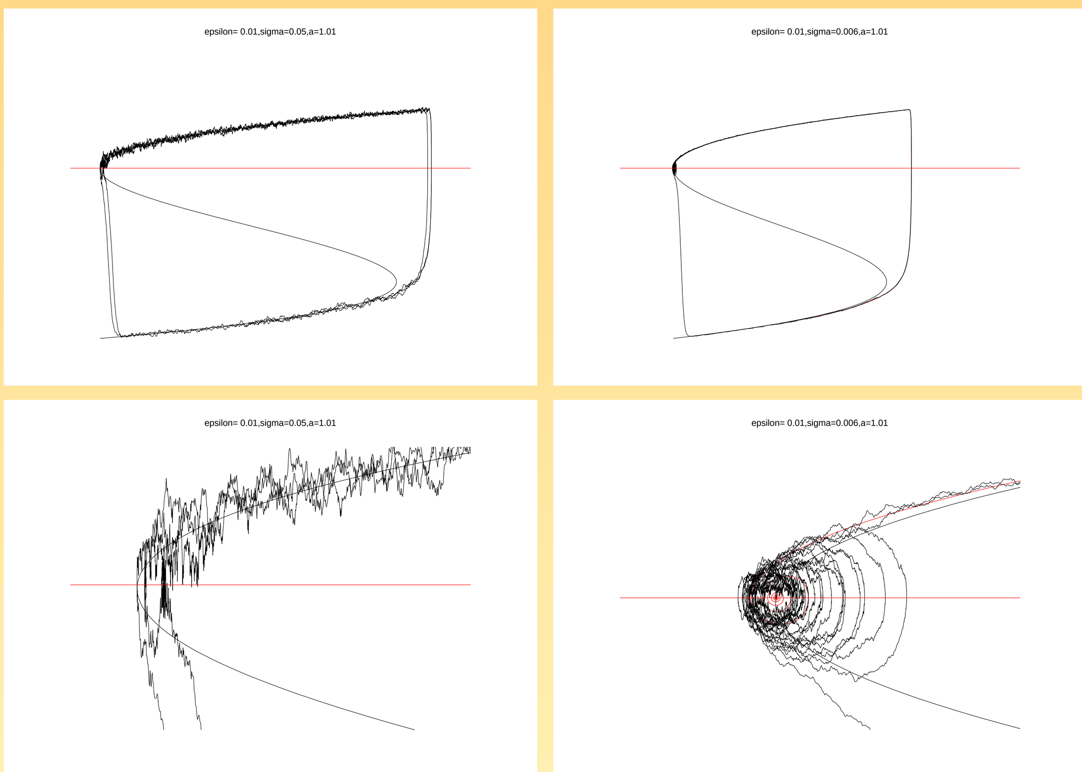
Now we add noise to the first line of the equation (1):

$$\begin{cases} \varepsilon dx_t = \left(x_t - \frac{x_t^3}{3} + y_t\right) dt + \sigma\sqrt{\varepsilon} dW_t \\ dy_t = (\alpha - x_t) dt \end{cases} \qquad (2)$$

Then we have four cases:

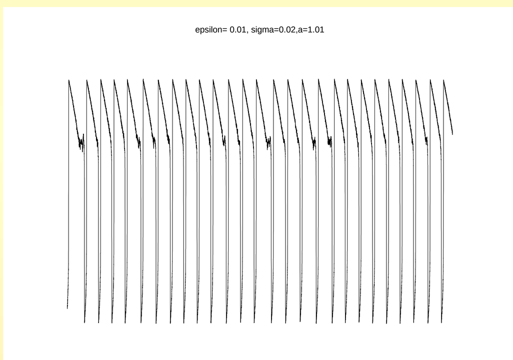
- node : **loops near the limit cycle**. There is no change to the global solution.
- focus :
  - weak noise : **loops around the fixed point**.
  - stronger noise : **loops around the fixed point and exit** (*right*) from the neighborhood of the fixed point and loop on the limit cycle.
  - strong noise : **loop near the limit cycle** (*left*).

$\sigma = 0.05$ $\sigma = 0.005$

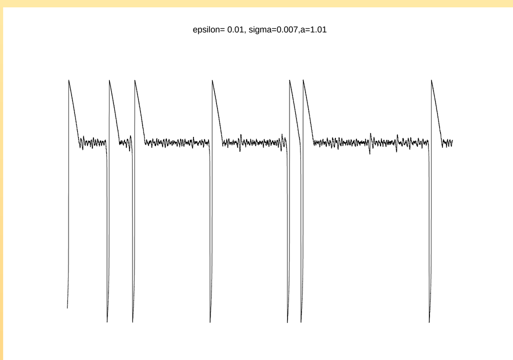


Finally, we fix α and we plot the membrane potential x in function of the time t. We observe three different main regimes following the value of σ (see [2]) :

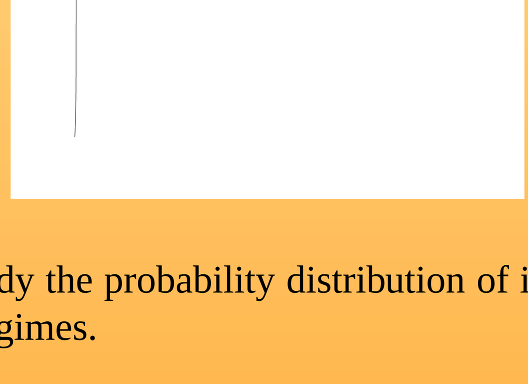
- **numerous and regular spikes** : the trajectory stay only a few times around the equilibrium point before exiting ($\sigma = 0.02$).



- **spike or a cluster of spikes** from time to time. That mean the trajectory stay some times around the equilibrium point before exiting and when it come back, it can sometimes exit quickly ($\sigma = 0.007$)



- **rare isolated spikes** ($\sigma = 0.001$).

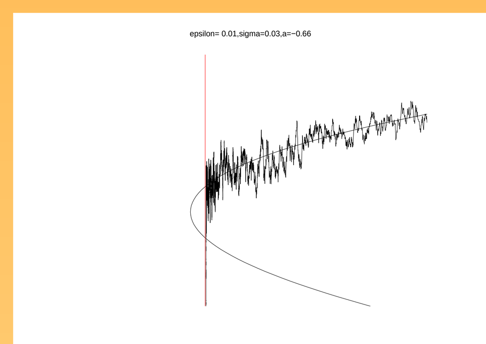


We want to study the probability distribution of inter-spikes time in this different regimes.

Exit time of potential well

In this part, we study the equation (FHN) with $\beta = 0$ and $\gamma = 1$:

$$\begin{cases} \varepsilon dx_t = \left(x_t - \frac{x_t^3}{3} + y_t\right) dt + \sqrt{\varepsilon}\sigma\, dW_t \\ dy_t = (\alpha - y_t) dt \end{cases} \qquad (3)$$



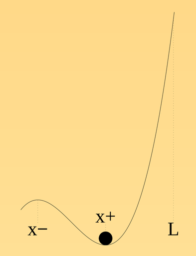
In the neighborhood of the Hopf bifurcation point, the equation (3) come down to the study of the equation:

$$dx_t = -\frac{1}{\varepsilon} V'(x_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

where V is the potential:

$$V(x) = -\delta x + \frac{1}{3} x + \gamma x^4$$

Here δ and γ are two real parameter which depend on α, x^* and y^*.



Potential V with the particular abscissas x_-,x_+ and L

We will study the exit times of the neighborhood of Hopf bifurcation (x^*, y^*). It is the same to study the exit times of the potential well (see [1]).

We define τ as the first time of exit of the potential well by abscissas x_- or L:

$$\tau = \inf\{t > 0 : x_t \in \{x_-, L\}\}.$$

L is a positive large real $(L >> 1)$.

We define the differential operator $\mathcal{L}$ by:

$$\mathcal{L} = -\frac{1}{\varepsilon} V'(x) \frac{d}{dx} + \frac{\sigma^2}{2\varepsilon} \frac{d^2}{dx^2}$$

and the problems

$$\begin{cases} (\mathcal{L} - \lambda) u^\lambda(x) = 0 \\ u^\lambda(x_-) = k \qquad k = 1, 0 \\ u^\lambda(L) = 0 \end{cases} \qquad \text{(PB)}$$

Using Feyman-Kac formula, we obtain

$$u_\lambda(x) = \mathbb{E}_x \left[e^{\lambda \tau_{x_-}} \mathbb{I}_{\tau_{x_-} < \tau_L} \right]$$

is solution of (PB) with k=1.

Proposition . *The density of τ_{x_-}, the time of exit by τ_{x_-}, follow* ***an asymptotically exponential law of parameter λ_1*** *which is the first eigenvalue of the problem (PB) with $k = 0$.*

PROOF : We remark that $u_\lambda(x)$ is Laplace transform of the density of τ_{x_-}. Thus, the density of τ_{x_-} is the inverse Laplace transform of $u_\lambda(x)$.

The problem (PB) with $k = 1$ has a solution if λ is not an eigenvalue of the problem with $k = 0$.

We can estimate the value of the two first eigenvalue and have an approximation of the eigenvalues λ_n as $n \to \infty$:

- $\lambda_1 = \frac{\sqrt{|V''(x_+)V''(x_-)|}}{\pi\varepsilon} \exp\left(-\frac{2}{\sigma^2}[V(x_-) - V(x_+)]\right) \left[1 + O\left(\frac{\exp\left(-2H/\sigma^2\right)}{\sigma^2}\right)\right]$ as $\sigma \to 0$
- $\lambda_2 \geq C\sigma^2$, as $\sigma \to 0$. Here C is a constant independent of σ.

As u_λ is a meromorphic function, to calculate the inverse Laplace transform, we have to estimate the residues of the function $\lambda \mapsto u^\lambda e^{-t\lambda}$ in the eigenvalues of the problem (PB) with $k = 0$:

$$\left(\mathfrak{L}^{-1} u^\lambda\right)(t) = \sum_{\lambda \in \{\lambda_1, \cdots\}} Res(u^\lambda e^{-t\lambda}, \lambda)$$

$$= \lambda_1 e^{-\lambda_1 t} [1 - \sigma^2 + O\left(\sigma |\log \sigma|^2\right)] \qquad \blacksquare$$

References

[1] N. BERGLUND and B. GENTZ Noise-Induces Phenomena in Slow-Fast Dynamical Systems. *Springer*, 2005.

[2] C. MURATOV and E. VANDEN-EIJENDEN. Noised-induced mixed-mode oscillations in a relaxation oscillator near the onset of a limit circle *Chaos*, 18, 2008.